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NOTE ON REGULAR POLYGONS*

By Charlotte Angas Scott

It is a well-known fact that the circle described with its centre at any point P of a rectangular hyperbola and passing through the other extremity of the diameter through P cuts the hyperbola again at the vertices of an equilateral triangle. The object of the present note is to show that the circles which cut a rectangular hyperbola at four of the vertices of a regular pentagon, heptagon, or nonagon can be described with equal facility.

It is necessary, in the first place, to find a rectangular hyperbola that shall cut a given circle, $x^2 + y^2 = 1$, at four vertices of one of these polygons.

1. The Heptagon. If a side of the heptagon subtends at the centre an angle θ, then $7\theta = 2\pi$, hence $\sin 3\theta = -\sin 4\theta$; that is,

$$3 \sin \theta - 4 \sin^3 \theta = -4 \sin \theta \cos \theta \ (2 \cos^2 \theta - 1),$$

from which,

$$\sin \theta \left\{ 8 \cos^3 \theta + 4 \cos^2 \theta - 4 \cos \theta - 1 \right\} = 0.$$

If one vertex is at (1, 0) (see figure 1) the roots of this equation give the values of θ for the vertices. If x is written for $\cos \theta$, the solution $\sin \theta = 0$, which refers to the vertex (1, 0) and gives therefore $\theta = 0$, becomes $x - 1 = 0$; and the cubic factor gives

$$8x^3 + 4x^2 - 4x - 1 = 0.$$

Hence the roots of

$$(x - 1)(8x^3 + 4x^2 - 4x - 1) = 0,$$

that is, of

$$8x^4 - 4x^3 - 8x^2 + 3x + 1 = 0,$$

are the abscissæ of the vertices numbered 7, 1 and 6, 2 and 5, 3 and 4, in figure 1.

It is desired to find a rectangular hyperbola that shall meet the circle at the point 7, and at one point of every pair (1, 6), (2, 5), (3, 4). There are clearly 2^3, that is, 8 such rectangular hyperbolas; any one will serve the purpose.

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* Presented to the American Mathematical Society, October 27, 1906.

The general equation of a rectangular hyperbola is

$$ax^2 + 2hxy - ay^2 + 2gx + 2fy + c = 0.$$

To find where this meets the circle

$$x^2 + y^2 - 1 = 0,$$

eliminate y. The two equations give, by elimination of the term y^2,

$$2ax^2 + 2hxy + 2gx + 2fy + c - a = 0,$$

that is,

$$2ax^2 + 2gx + c - a = -y\ (2hx + 2f)\ ;$$

hence, since

$$-y^2 = x^2 - 1,$$

$$(x^2 - 1)(2hx + 2f)^2 + (2ax^2 + 2gx + c - a)^2 = 0.$$

The equation for the abscissæ of the common points is therefore

$$(4h^2 + 4a^2)\,x^4 + (8hf + 8ag)\,x^3 + \{4f^2 - 4h^2 + 4g^2 + 4a(c-a)\}\,x^2 \\ + \{-8hf + 4g(c-a)\}\,x - 4f^2 + (c-a)^2 = 0,$$

and this is to be the same as

$$8x^4 - 4x^3 - 8x^2 + 3x + 1 = 0.$$

Hence, equating coefficients,

$$4h^2 + 4a^2 = 8\lambda, \tag{1}$$

$$8hf + 8ag = -4\lambda, \tag{2}$$

$$4f^2 - 4h^2 + 4g^2 + 4a(c-a) = -8\lambda, \tag{3}$$

$$-8hf + 4g(c-a) = 3\lambda, \tag{4}$$

$$-4f^2 + (c-a)^2 = \lambda. \tag{5}$$

The sum of these gives

$$4g^2 + 4g(c+a) + (c+a)^2 = 0,$$

that is,

$$2g + c + a = 0.$$

The sum of (2) and (4) gives

$$4g(c + a) = -\lambda.$$

Hence $\qquad -2g = a + c, \qquad \lambda = 2(a + c)^2.$

With these values for g and λ, (1), (5), (2) become

$$h^2 = 4(a + c)^2 - a^2,$$

$$4f^2 = (c - a)^2 - 2(a + c)^2 = -(a + c)^2 - 4ac,$$

$$2hf = a(a + c) - 2(a + c)^2.$$

Hence, in virtue of the identity $h^2 \times 4f^2 = (2hf)^2$, we have

$$\{4(a + c)^2 - a^2\}\{-(a + c)^2 - 4ac\} = (a + c)^2\{a - 2(a + c)\}^2.$$

Since $2(a + c) - a$ is a factor of each side of this equation, one solution is $2(a + c) - a = 0$, that is,

$$a = -2c.$$

This leads to

$$\begin{aligned} 2g &= -(a + c) = c, \\ \lambda &= 2(a + c)^2 = 2c^2, \\ h^2 &= 4(a + c)^2 - a^2 = 4c^2 - 4c^2 = 0, \\ 4f^2 &= -(a + c)^2 - 4ac = -c^2 + 8c^2 = 7c^2. \end{aligned}$$

Since c is a factor in all the coefficients, it is not zero; hence we may give it any convenient value, for example, -1. Then $a = 2$, $h = 0$, $2g = -1$, $2f = \pm\sqrt{7}$; choose for $2f$ the value $+\sqrt{7}$.

The four possible values for $a : c$, and the two for f, account for the eight solutions of the problem. The other three values for $a : c$, given by the cubic

$$\{2(a + c) + a\}\{-(a + c)^2 - 4ac\} = (a + c)^2\{2(a + c) - a\},$$

are irrational. Since only one solution out of the eight is needed, it is unnecessary to attend to these.

The rectangular hyperbola is

$$2x^2 - 2y^2 - x + \sqrt{7}\,y - 1 = 0.$$

The center is at $(\frac{1}{4}, \frac{1}{4}\sqrt{7})$; when the origin is transferred to this point the equation becomes

$$x^2 - y^2 = \frac{1}{8},$$

hence the semi-axis major is $\frac{1}{4}\sqrt{2}$. The curve lies as shown in figure 1 ; it passes through the vertices 1, 2, 4, 7.

Since the foci of $x^2 - y^2 = \frac{1}{8}$ are $(\pm \frac{1}{2}, 0)$, with the original axes they are $(\frac{3}{4}, \frac{1}{4}\sqrt{7})$, $(-\frac{1}{4}, \frac{1}{4}\sqrt{7})$; of these the first lies on the circle. The corresponding directrices are $x = \pm \frac{1}{4}$, that is, referred to the original axes, $x = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$, and $x = -\frac{1}{4} + \frac{1}{4} = 0$; the second of these passes through the centre of the circle. The distance between the foci is equal to the radius of the circle.

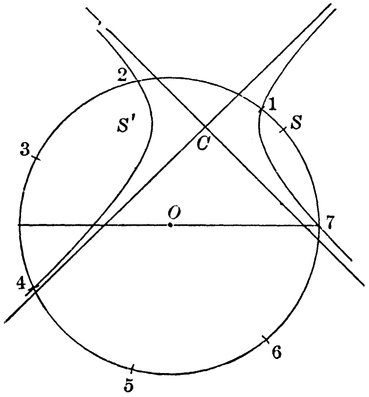

FIG. 1.

2. The Nonagon. In the case of the nonagon, the angle θ is determined by $3\theta = \frac{2}{3}\pi$; hence $\cos 3\theta = -\frac{1}{2}$, that is,

$$4\cos^3\theta - 3\cos\theta + \tfrac{1}{2} = 0,$$

or

$$8x^3 - 6x + 1 = 0.$$

The roots of this equation are the abscissae of the vertices 1, 4, 7 (see figure 2), hence of the pairs of vertices (1, 8), (4, 5), (7, 2). To bring in 3 and 6, for which $x = -\frac{1}{2}$, we multiply by $2x + 1$; the equation becomes

$$16x^4 + 8x^3 - 12x^2 - 4x + 1 = 0.$$

By the same argument as in the case of the heptagon, the coefficients in the equation of the desired rectangular hyperbola are determined by the five equations

$$4h^2 + 4a^2 = 16\lambda, \qquad (1)$$

$$8hf + 8ag = 8\lambda, \qquad (2)$$

$$4f^2 - 4h^2 + 4g^2 + 4a(c-a) = -12\lambda, \qquad (3)$$

$$-8hf + 4g(c-a) = -4\lambda, \qquad (4)$$

$$-4f^2 + (c-a)^2 = \lambda. \qquad (5)$$

These, combined exactly as before, give

$$4g^2 + 4g(c+a) + (c+a)^2 = 9\lambda,$$

$$4g(c+a) = 4\lambda,$$

hence $$4g^2 - 5g(c+a) + (c+a)^2 = 0,$$

that is, $$\{4g - (c+a)\}\{g - (c+a)\} = 0$$

from which $$4g = c + a \quad \text{or} \quad g = c + a.$$

Of these, $4g = c + a$ leads to the simplest results; we find $4\lambda = (c+a)^2$, and then equations (1), (5), (2) become

$$h^2 = (a+c)^2 - a^2 = c(2a+c),$$

$$16f^2 = 4(c-a)^2 - (a+c)^2 = (a-3c)(3a-c),$$

$$4hf = (a+c)^2 - a(a+c) = c(a+c).$$

Hence, by means of $h^2 \times 16f^2 = (4hf)^2$,

we have

$$c(2a+c)(a-3c)(3a-c) = c^2(a+c)^2.$$

One solution of this equation is $c = 0$; the other three are irrational. We take therefore $c = 0$; then $h = 0$, $4g = a$, $16f^2 = 3a^2$, from which $4f = \pm\sqrt{3}\,a$; take $4f = +\sqrt{3}\,a$.

(Since there are four pairs of vertices, the number of solutions to be expected is 2^4, that is 16. All are accounted for; two values for g, four for $a : c$, two for f, give $2 \times 4 \times 2 = 16$ solutions).

Since a is a factor in all the coefficients, it is not zero; hence we may assign any convenient value, in this case 2. The desired rectangular hyperbola is

$$2x^2 - 2y^2 + x + \sqrt{3}\,y = 0.$$

The centre is at $(-\frac{1}{4}, \frac{1}{4}\sqrt{3})$; when the origin is transferred to this point the equation becomes

$$-x^2 + y^2 = \frac{1}{8},$$

hence the semi-axis major is $\frac{1}{4}\sqrt{2}$. The curve lies as shown in figure 2; it passes through the vertices 2, 3, 5, 8, and through the centre of the circle.

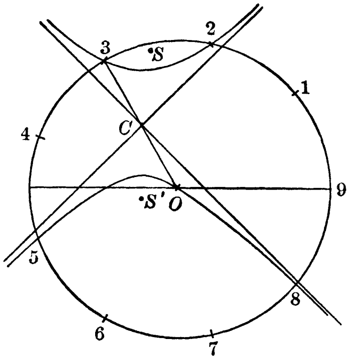

FIG. 2.

The radius of the circle to the vertex 3 is a diameter of the hyperbola; the distance between the foci is equal to the radius of the circle.

The rectangular hyperbola thus obtained for the nonagon is of the same size as that obtained for the heptagon, but differently placed.

3. The Pentagon. In the case of the pentagon, $5\theta = 2\pi$, hence $\sin 2\theta = -\sin 3\theta$,

from which

$$2 \sin \theta \cos \theta + 3 \sin \theta - 4 \sin^3 \theta = 0;$$

that is,

$$\sin \theta = 0 \quad \text{or} \quad 2 \cos \theta + 3 - 4 \sin^2\theta = 0.$$

Hence

$$4 \cos^2\theta + 2 \cos\theta - 1 = 0,$$

that is,

$$4x^2 + 2x - 1 = 0.$$

The roots of this quadratic are the abscissæ of the pairs of vertices (1, 4), (2, 3) see figure 3. A rectangular hyperbola, with the axis of x as one axis of symmetry, can be passed through the four points. The equation of such a hyperbola is

$$ax^2 - ay^2 + 2gx + c = 0;$$

the abscissæ of the intersections with the circle are the roots of

$$ax^2 + a(x^2 - 1) + 2gx + c = 0,$$

that is, of

$$2ax^2 + 2gx - (a - c) = 0.$$

Hence $2a = 4$, $2g = 2$, $a - c = 1$, and the hyperbola is therefore

$$2x^2 - 2y^2 + 2x + 1 = 0.$$

The centre is at $(-\frac{1}{2}, 0)$; when this point is taken as origin the equation becomes

$$x^2 - y^2 + \tfrac{1}{4} = 0,$$

hence the semi-axis major is $\frac{1}{2}$. The curve lies as shown in figure 3. The radius of the circle to the point $(-1, 0)$ is the minor axis of the hyperbola.

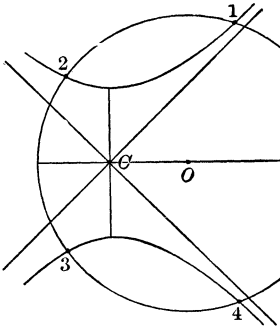

FIG. 3.

4. Conclusion. If we change the origin in all three cases to the centre of the hyperbola, and change the scale so that the hyperbola has the semi-axis major $= 1$; and moreover, in the case of the heptagon, interchange the axes of x and y; we arrive at the conclusion that the hyperbola $x^2 - y^2 + 1 = 0$ is cut at four vertices of a regular pentagon by the circle $(x - 1)^2 + y^2 = 4$; at four vertices of a regular heptagon by the circle

$$\left(x - \sqrt{\frac{7}{2}}\right)^2 + \left(y + \frac{1}{\sqrt{2}}\right)^2 = 8;$$

at four vertices of a regular nonagon by the circle

$$\left(x - \frac{1}{\sqrt{2}}\right)^2 + \left(y + \sqrt{\frac{3}{2}}\right)^2 = 8.$$

The relation of the different circles to the hyperbola is, however, more neatly expressed by the geometrical statements of the results :

(1) A rectangular hyperbola is cut at four vertices of a regular pentagon by a circle on the minor axis as a radius (figure 4).

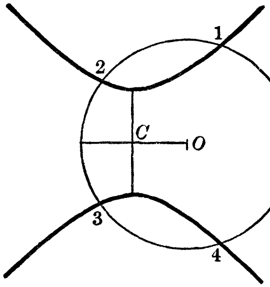

Fig. 4.

(2) A rectangular hyperbola is cut at four vertices of a regular heptagon by a circle of radius equal to the distance between the foci, with its centre on one directrix and passing through the other focus (figure 5).

(3) A rectangular hyperbola is cut at four vertices of a regular nonagon by a circle which has for radius a diameter of the hyperbola of length equal to the distance between the foci (figure 6).

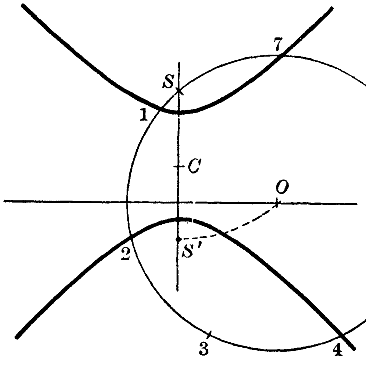

Fig. 5.

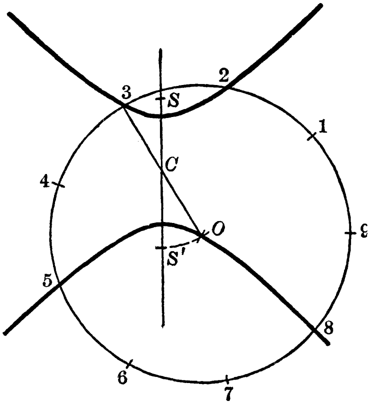

Fig. 6.

Bryn Mawr College,
Bryn Mawr, Pennsylvania,
October, 1906.